Nonlinear Tax Structures and Endogenous Growth

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Abstract

This paper investigates the growth implications of a nonlinear tax structure. The interest here is on the distortionary not the redistributive effects of taxation on economic growth. The paper finds two results. First, the inclusion of a nonlinear tax structure into an AK growth model introduces the convergence behavior of the neoclassical growth model, while retaining the steady-state growth properties of the AK model. Second, a tax structure that is more-progressive through time will lower the transitional growth rate and raise the speed of convergence. Therefore, this paper suggests that the tax structure may be another source of observed differences in per capita growth rates.

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I. INTRODUCTION

During the past few years, there has been a rapid expansion in research that explores the growth implications of tax policy. Papers by Lucas (1990), Jones and Manuelli (1990), King and Rebelo (1990), Jones, Manuelli and Rossi (1993) and others argue that increased taxation reduces long-run economic growth. Using a variety of endogenous growth models, these researchers show that increased taxation leads to greater distortions in the economy and thus lowers the steady-state growth rate. However, with few exceptions, the tax policy that is considered is a constant, flat-rate tax. As a result, these papers speak little about the effect of tax structure on long-run economic growth.

In this paper, the growth implications of a nonlinear tax structure are investigated. As in the above-mentioned studies, the interest here is on the distortionary not the redistributive effects of taxation on economic growth. Consequently, a representative agent framework is adopted to focus attention on the agent’s decisions when future tax rates are connected to current decisions. The paper finds two results. First, the inclusion of a nonlinear tax structure into an $\text{Ak}$ growth model introduces the convergence behavior of the neoclassical growth model, while retaining the steady-state growth properties of the $\text{Ak}$ model. Second, economies whose tax structure is more-progressive through time will have a faster speed of convergence and experience lower transitional growth rates than those economies whose tax structure is less-progressive through time. A tax structure is considered more-progressive through time if the ratio of the marginal to average tax rate with respect to aggregate income is higher.
In a standard $Ak$ model, there are no transitional dynamics.\textsuperscript{1} As a result, a one-time exogenous increase in a flat-rate tax immediately lowers the steady-state growth rate. In the model presented here, however, the average and marginal tax rates are a positive and concave function of income. Therefore, as the capital stock and thus the level of income increases, the average and marginal tax rates rise endogenously. In response, the after-tax return on capital falls, which leads to reductions in the growth rates of capital, consumption and output. This transition process continues until the average and marginal tax rates converge to an upper bound. As a result, this model exhibits the convergence property of falling per capita growth rates as seen in the neoclassical growth model.

The second result is that economies whose tax structure is more-progressive through time will have a faster speed of convergence and thus experience lower transitional growth rates. Consider two economies -- X and Z -- beginning with the same initial stock of capital $k(0)$. Each economy is represented by one agent whose return on both capital and labor are taxed. Tax revenues are transferred back to the households and consumed by the government. Both economies are identical -- including their steady state or upper tax rates -- except that economy X has a tax structure that is more-progressive through time. As a result, both economies possess the same steady-state growth rate. At time=0, the agent in economy X faces a higher marginal tax rate on $k(0)$ than his counterpart in economy Z. In response, the agent in economy X consumes more of $k(0)$ which leads to lower transitional growth rates of capital, output and consumption. Furthermore, due to a higher degree of progressivity, the

\textsuperscript{1} See chapter 4 of Barro and Sala-i-Martin (1995). One exception is the one-sector, growth model of Jones and Manuelli (1990) that also possesses both steady-state growth and transitional dynamics.
growth rate of the average tax burden is higher in economy X. Therefore, tax rates and thus
growth rates in economy X converge faster to steady state than those in economy Z.

The rest of the paper proceeds as follows. Section II describes and then solves an Ak
model with a nonlinear tax function. Section III determines the conditions for steady-state
growth. Section IV posits a functional form for the tax function. Section V derives the
transitional dynamics of the model. Section VI discusses the implications of including
heterogeneous agents. Section VII concludes.

II. THE MODEL

The model adopted follows the Ak model of Rebelo (1991). The representative agent chooses
a consumption path \( c(t) \) and asset accumulation \( \dot{a}(t) \) to maximize the stream of discounted
utility:

\[
U = \int_{t=0}^{\infty} e^{-\rho t} \cdot (1-\theta)^{-1} \cdot [c(t)^{(1-\theta)} - 1] \cdot dt \quad \theta > 0, \quad \rho \leq 1
\]  

(1)

subject to

\[
\dot{a}(t) \leq r(t) \cdot a(t) + b(t) - \tau(y) - c(t)
\]  

(2)

where \((1/\theta)\) is the elasticity of intertemporal substitution, \(\rho\) is the subjective discount rate,
\(a(t)\) is the stock of claims on physical and human capital, \(r(t)\) is the interest rate paid on those
claims, \(b(t)\) are government transfer payments and \(\tau(y)\) is total tax paid.

Households take the interest rate \(r(t)\) and the initial stock of wealth \(a(0)\) as given.
However, since total tax paid is a positive function of income, household asset decisions do
affect marginal and average tax rates. As a result, households take only the parameters of the
tax function \(\tau(y)\) and not the tax rates themselves as given.
Let $\lambda(t)$ represent the current-value shadow price of wealth. The optimal conditions are

$$
\lambda(t) = e^{-\rho} \tag{3}
$$

$$
\dot{\lambda}(t) = \lambda(t) \cdot [\rho - r(t) \cdot (1 - \tau'(y))] \tag{4}
$$

$$
\lim_{t \to \infty} \{ a(t) \cdot e^{-\rho} \cdot \lambda(t) \} = 0 \tag{5}
$$

where $\tau'(y)$ represents the marginal tax rate -- derivative of total tax paid with respect to total income $y(t)$. Equations (3) and (4) in conjunction with the budget constraint (2) specify the optimal paths of consumption and wealth, while the transversality condition (5) specifies the terminal condition. Notice, however, that along these optimal paths both the average and marginal tax rates increase endogenously.

Firms produce a single consumption good $y$ using the linear production function:

$$
y(t) = A \cdot k(t), \quad A > 0 \tag{6}
$$

where $k(t)$ is the stock of both human and physical capital. Capital is assumed to depreciate at the constant rate $\delta$. Zero profit condition requires that $r(t) = A - \delta$ for all $t$.

Lastly, the government collects taxes $\tau(y)$, transfers $b(t)$ of the receipts back to households and consumes the remainder. This government consumption does not directly contribute to either production or utility. The government sets the transfer payments so that a fixed percentage of tax revenues are rebatted back to the household

$$
b(t) = \xi \cdot \tau(y) \tag{7}
$$

where $\xi \in [0,1]$. The restrictions on the tax function $\tau$ are as follows:

\footnote{If government consumption entered the utility function (1) in an additively-separable way, private consumption decisions would remain unaffected by government consumption.}
\[ \tau'(y), \tau''(y) > 0 \]  \hspace{1cm} (8)

\[ \frac{\tau(y)}{y(t)} > 0 \]  \hspace{1cm} (9)

\[ \lim_{y \to 0} \tau'(y) = \lim_{y \to 0} \left[ \frac{\tau(y)}{y(t)} \right] = \bar{\tau} \geq 0 \]  \hspace{1cm} (10)

\[ \lim_{y \to 0} \tau'(y) = \lim_{y \to 0} \left[ \frac{\tau(y)}{y(t)} \right] = \bar{\tau} > 0 \]  \hspace{1cm} (11)

where \( \bar{\tau} > \bar{\tau} \). With total tax paid assumed to be a convex function of taxable income \( y(t) \), the marginal tax rate \( \tau'(y) \) exceeds the average tax rate \( \tau(y)/y(t) \), and thus the tax system is considered progressive.\(^3\) This type of progressivity will be referred to as progressivity through time since it describes the relationship between tax rates and aggregate income.

Conditions (10) and (11) imply that the two tax rates converge to either a common lower or upper bound.

Substituting (3), (6), (7) and the equilibrium conditions into (2) and (4), the following two differential equations govern the evolution of consumption and capital

\[ \frac{\dot{c}(t)}{c(t)} = \left( 1/\theta \right) \cdot \left[ A \cdot (1 - \tau'(y)) - \delta - \rho \right] \]  \hspace{1cm} (12)

\[ \frac{\dot{k}(t)}{k(t)} = A \cdot \left[ 1 - (1-\xi) \cdot (\tau(y)/y(t)) \right] - \delta - \left[ \frac{c(t)}{k(t)} \right] \]  \hspace{1cm} (13)

given the initial stock of capital \( k(0) \) and the transversality condition (5). According to the Euler equation (12), growth in consumption occurs so long as the after-tax marginal product of capital exceeds the discount rate. According to the capital accumulation equation (13),

\[ \text{(3) Even though there is no single “correct” way to measure the degree of progression, Musgrave and Musgrave (1989) suggest three measures of progression: average-rate} \frac{\dot{c}[\tau(y)/y]}{\dot{\theta}y}; \text{ liability } \frac{\tau'(y)/[\tau(y)/y]}{[\tau'(y)/y]}; \text{ and residual income } (\frac{\dot{c}[\tau(y)/y]}{\dot{\theta}y})/[(y - \tau(y))/y]. \text{ A tax system is considered progressive if average-rate progression is greater than zero, liability progression is greater than one, and residual income progression is less than one. It is relatively straightforward to show that (8)-(11) satisfies all three measures of progression.} \]
growth in the stock of capital occurs so long as the net average product of capital on non-transferred income exceeds the consumption to capital ratio. Note that if all tax revenues are transferred back ($\xi = 1$) then the average tax rate drops out of (13). The difference between this set of equations and those in the standard $Ak$ model is that the marginal and average tax rates depend upon the state variable $k(t)$.

III. STEADY STATE

Define steady state as a situation in which $c(t)$, $k(t)$ and $y(t)$ grow at constant rates. Appendix A shows that all three variables must grow at the common rate of $\gamma^*$ in steady state. The necessary and sufficient conditions for steady-state growth is:

$$
\frac{(\rho + \delta)}{(1 - \bar{r})} < A < \frac{[\rho + \delta \cdot (1 - \bar{\theta})]}{[(1 - \bar{r}) \cdot (1 - \bar{\theta})]}.
$$

(14)

See Appendix A for the proof. If the first inequality holds, then the after-tax marginal product of capital would exceed the discount rate. According to the Euler and capital accumulation equations, steady-state growth in consumption, capital and output would occur. If the second inequality holds, then the after-tax marginal product of capital would grow faster than the stock of capital and the transversality condition would hold.

The steady-state growth rate of consumption, capital and output under condition (14) is

$$
\gamma^* = \frac{1}{(1/\theta)} \cdot \left[A \cdot (1 - \bar{r}) - \delta - \rho\right].
$$

(15)

Moreover, the model exhibits the convergence property of declining per capita growth rates during the transitional period. As the level of income increases, average and marginal tax rates rise which leads to decreases in the after-tax return on capital. This causes transitional growth rates to fall until tax rates converge to their upper limit $\bar{r}$. Therefore, transitional growth rates will depend upon the speed at which tax rates converge to $\bar{r}$. 

IV. TAX FUNCTION

To explicitly derive these transitional dynamics, a functional form for the tax function is posited. The functional form must satisfy conditions (8)-(11) and also produce a linear relationship between the average and marginal tax rate. This linear relationship will allow (12) and (13) to be transformed into variables that are constant in steady state. The following tax function satisfies these conditions:

\[
\tau(y) = \tau_0 \cdot y(t) - \tau_1 \cdot y(t)^n \quad \text{for } y(t) \geq \left( \frac{\tau_1}{\tau_0} \right)^{\frac{1}{n-1}}
\]

\[
\tau(y) = 0 \quad \text{for } y(t) < \left( \frac{\tau_1}{\tau_0} \right)^{\frac{1}{n-1}}
\]

where

\[
\tau'(y) = \tau_0 - \tau_1 \cdot n \cdot y(t)^{n-1} \quad \text{for } y(t) \geq \left( \frac{\tau_1}{\tau_0} \right)^{\frac{1}{n-1}}
\]

\[
\frac{\tau(y)}{y(t)} = \tau_0 - y(t)^{n-1} \quad \text{for } y(t) \geq \left( \frac{\tau_1}{\tau_0} \right)^{\frac{1}{n-1}}.
\]

The assumptions on the parameters are that \(0 < \tau_0 < 1, \tau_1 > 0\) and \(0 < n < 1\). Equations (17) and (18) describe the marginal and average tax rates, respectively. The linear relationship between the two tax rates is \(\tau(y)/y(t) = \lfloor \tau'(y)/n \rfloor + \tau_0 - \lfloor \tau_0/n \rfloor\).

The tax function (16) has the following properties. First, the tax function is considered progressive since the marginal tax rate exceeds the average for finite \(y(t)\). The degree of progression is captured by the parameter \(n\) where a decrease in \(n\) makes the tax function more progressive through time. Second, as \(y(t) \to \infty\), then both the marginal and average tax rate converge to a common upper tax rate \(\tau_0\). This property of converging tax rates is also found in modern developments of the principle of equal sacrifice in Young (1988) (1990) and
Berliant and Gouveia (1993). Third, given values for \( n \) and \( \tau_0 \), the parameter \( \tau_1 \) determines the initial level of taxable income.

V. TRANSITIONAL DYNAMICS

To describe the transitional dynamics, equations (12) and (13) must be transformed into variables that are constant in steady state. Let \( z(t) \) represent the after-tax output to capital ratio \( [y(t) - \tau(t)] / k(t) \) and \( \chi(t) \) represent the consumption to capital ratio \( c(t)/k(t) \). The \( z \) variable becomes the state-like variable and the \( \chi \) variable, which can jump discretely, becomes the control-like variable. As a result, the evolution of the economy is described as follows:

\[
\dot{z} = -\left[ z \cdot (1 - \xi) - \chi + \xi \cdot A - \delta \right] \cdot \left[ (1 - n) \cdot (z \cdot A - (1 - \tau_0)) \right] \tag{19}
\]

\[
\dot{\chi} = \chi \cdot \left[ (\chi - \phi) - \left[ (\theta \cdot (1 - \xi) - n) / \theta \right] \cdot (z \cdot A - (1 - \tau_0)) \right] \tag{20}
\]

where

\[
\phi \equiv (1/\theta) \cdot \left[ (\theta - 1) \cdot (A \cdot (1 - \tau_0) - \delta) + \theta \cdot \xi \cdot A \cdot \tau_0 + \rho \right]. \tag{21}
\]

Note that the time subscripts have been dropped temporarily for clarity. In (19), the \([\]\) term represents the growth rate of capital, while the \(\{}\) term represents tax progressivity \( A \cdot (r'(y) - \tau(y) / y(t)) \). An increase in \((1 - n)\) raises the value of the \(\{}\) term and ultimately the speed of convergence. It is important to note that this positive relationship between \( \dot{z} \) and \((1 - n)\) is driven by the change in tax progressivity and not the change in tax incidence.\(^4\) In

\(^4\) This claim can be supported by considering the alternative function: \( \tau(y) = \tau_0 \cdot y(t) - (\tau_1 / n) \cdot y(t)^n \). For this tax function, an increase in \((1 - n)\) makes the tax system more progressive by simultaneously raising the marginal tax rate and lowering the average tax rate when \( n \) and \( y(t) \) are low. However, under this alternative tax function, equation (20) is unchanged and thus the positive relationship between \( \dot{z} \) and \((1 - n)\) remains.
(20), the \{ \} term represents the difference between the growth rate of consumption and capital. Lastly, since condition (14) is assumed to hold, $\phi \in (0, A \cdot (1 - \tau_0) - \delta + \xi \cdot A)$.

A. Phase Diagram

Figure 1 presents the phase diagram of (19) and (20) in $(z, \chi)$ space. Equation (19) implies that $\dot{z} = 0$ if $\chi = A \cdot (1 - \tau_0)$ or if $\chi \cdot (1 - \xi) = \chi - \xi \cdot A + \delta$. The first condition shows up as a vertical line drawn at $A \cdot (1 - \tau_0)$, while the second condition shows up as the line $\chi = z \cdot (1 - \xi) + \xi \cdot A - \delta$. These two lines intersect where $\chi = A \cdot (1 - \tau_0) + A \cdot \xi \cdot \tau_0 - \delta$. If $z$ is greater than both $A \cdot (1 - \tau_0)$ and $(\chi + \delta - \xi \cdot A)/(1 - \xi)$, then $\dot{z} < 0$ and $z$ will fall. However, if $z$ is greater than $A \cdot (1 - \tau_0)$ but less than $(\chi + \delta - \xi \cdot A)/(1 - \xi)$, then $\dot{z} > 0$ and $z$ will rise. Since $z \geq A \cdot (1 - \tau_0)$ by definition, all values of $z$ strictly less than $A \cdot (1 - \tau_0)$ are irrelevant.

Equation (20) implies that $\dot{\chi} = 0$ if $\chi = \phi - [(\theta \cdot (1 - \xi) - n)/\theta] \cdot A \cdot (1 - \tau_0) + [(\theta \cdot (1 - \xi) - n)/\theta] \cdot z$. The slope of the $\dot{\chi} = 0$ locus depends upon the values of $\theta$, $\xi$ and $n$.

Figure 1 is drawn assuming that $\theta \cdot (1 - \xi) > n$. If $\chi$ is above the $\dot{\chi} = 0$ locus, then $\dot{\chi} > 0$ and $\chi$ will rise over time. However, if $\chi$ is below the $\dot{\chi} = 0$ locus, then $\dot{\chi} < 0$ and $\chi$ will fall.

B. Steady State

Steady state is defined where $\dot{\chi} = \dot{z} = 0$. The $\dot{\chi} = 0$ locus intersects the first $\dot{z} = 0$ locus -- the vertical line -- at $\chi^* = \phi$ and $z^* = A \cdot (1 - \tau_0)$. Since condition (14) holds, this intersection occurs below the intersection of the two $\dot{z} = 0$ loci. Likewise, the $\dot{\chi} = 0$ locus intersects the second $\dot{z} = 0$ locus at $z = A \cdot (1 - \tau_0) - (1/n) \cdot [A \cdot (1 - \tau_0) - \delta - \rho]$ and $\chi = z \cdot (1 - \xi) + \xi \cdot A - \delta$. Condition (14) implies that this intersection yields a value for $z$ less
than \( A \cdot (1 - \tau_0) \) which cannot occur. Therefore, the only steady-state equilibrium is \( z^* = A \cdot (1 - \tau_0) \) and \( \chi^* = \phi \). The steady-state growth rate for consumption, capital and output is

\[
\gamma^* = \frac{1}{\theta} \left( A \cdot (1 - \tau_0) - \delta - \rho \right). \tag{22}
\]

The growth rate (22) is identical to (15) except that \( \bar{\tau} \) has been replaced by \( \tau_0 \).

**C. Local Dynamics**

From figure 1, it is clear that the steady-state equilibrium is a saddle-point where the line \( \chi(z) \) is the stable arm. Appendix B linearizes (19) and (20) around the steady state \((z^*, \chi^*)\). The local dynamics of \( z \) and \( \chi \) are given by

\[
z(t) - z^* = [z(0) - z^*] \cdot \exp(\beta_1 \cdot t) = \tau_1 \cdot A^n \cdot k(0)^{n-1} \cdot \exp(\beta_1 \cdot t) \tag{23}
\]

\[
\chi(t) - \chi^* = [\chi(0) - \chi^*] \cdot \exp(\beta_1 \cdot t) = \tau_1 \cdot A^n \cdot k(0)^{n-1} \cdot \phi \cdot \left( \frac{\theta \cdot (1 - \xi) - n}{\theta \cdot (\phi - \beta_1)} \right) \cdot \exp(\beta_1 \cdot t) \tag{24}
\]

where \( \beta_1 \) is the negative root of the characteristic equation. The value of \( \beta_1 \) is given by

\[
\beta_1 \equiv - (1 - n) \cdot \gamma^*. \tag{25}
\]

The absolute value of \( \beta_1 \) is the local speed of convergence.

Along the stable arm \( \chi(z) \), the after-tax output to capital ratio \( z(t) \) falls. Total tax paid (16) is an increasing function of income. Therefore, the growth rate of after-tax output falls relative to the growth rate of capital. This causes \( z(t) \) to decrease during the transition.

However, the transition of the consumption to capital ratio \( \chi \) depends upon the values for \( \theta \cdot (1 - \xi) \) and \( n \). If \( \theta \cdot (1 - \xi) > n \), then \( \chi(0) \) exceeds \( \chi^* \) and \( \chi \) declines monotonically as in figure 1. On the other hand, if \( \theta \cdot (1 - \xi) < n \), then \( \chi(0) \) is below \( \chi^* \) and \( \chi \) rises monotonically.
There are four fiscal parameters \{ \tau_0, \tau_1, n, \xi \}. The steady-state growth rate \( \gamma^* \) is a negative function of the upper tax rate \( \tau_0 \). The parameters \( \tau_1 \) and \((1-n)\) enter only during the transition, while transfers \( \xi \) affect \( \chi^* \) but not \( \gamma^* \). However, the local rate of convergence \( |\beta| \) depends negatively upon the upper tax rate \( \tau_0 \) and positively upon the degree of progressivity \((1-n)\). As appendix B shows, \( |\beta| \) is determined by the speed at which \( z(t) \) approaches \( z^* \) in equation (19). In the neighborhood of steady state, the \[ \] term in (19) equals the steady-state growth rate. Therefore, an increase in \( \tau_0 \) lowers \( \gamma^* \) which causes the economy to converge slower to its steady state. Similarly, an increase in \((1-n)\) raises the tax progressivity \{ \} term in (19) and unambiguously raises the local rate of convergence.

**D. Transitional Growth Rates**

Appendix B solves the transitional growth rates in terms of the exogenous parameters:

\[
\gamma_c(t) = \gamma^* + \tau_1 \cdot A^n \cdot k(0)^{n-1} \cdot [1/\theta] \cdot \exp(\beta_1 \cdot t) \tag{26}
\]

\[
\gamma_y(t) = \gamma^* + \tau_1 \cdot A^n \cdot k(0)^{n-1} \cdot \left( \frac{\phi \cdot n - \theta \cdot (1-\xi) \cdot \beta_1}{\theta \cdot (\phi - \beta_1)} \right) \cdot \exp(\beta_1 \cdot t). \tag{27}
\]

Since the \[ \] terms in both (26) and (27) are positive, the three growth rates decline monotonically at the rate \( |\beta| \). Therefore, as in the neoclassical growth model, the model exhibits the convergence property of falling transitional growth rates and predicts that economies with lower values of \( k(0) \) grow faster. Moreover, since condition (14) holds, the model allows for steady-state growth as in the standard \( Ak \) model.

What are the transitional growth effects of changes in the four fiscal parameters? An increase in the upper tax rate \( \tau_0 \) has an indeterminant effect upon \( \gamma_y(t) \) since it lowers both \( \gamma^* \)
and the rate of convergence. Likewise, an increase in the percentage of tax revenue transferred back $\xi$ has an indeterminant effect. If the elasticity of intertemporal substitution is low, then households would consume a greater percentage of income today and thus accumulate less income tomorrow. However, if the elasticity is high, then households would save a greater percentage which would raise $\gamma_1(t)$. An increase in the initial level of taxable income $\tau_1$ lowers the marginal tax rate for each level of income. As a result, the transitional growth rate would rise in response. Lastly, an increase in the degree of progressivity $(1 - n)$ raises the speed of convergence, but has an indeterminant effect on the initial growth rate of the economy. As appendix C shows, the first effect dominates so long as $t \geq 1$ and $k(0)$ is not too small. Therefore, under most conditions, an increase in the degree of progressivity through time lowers the transitional growth rate of the economy.

VI. HETEROGENEOUS AGENTS

Easterly and Rebelo (1993) include heterogeneous agents into an $Ak$ growth model with a nonlinear tax schedule. Because of the complexities introduced, they could not solve for the growth rate of income. However, they found that the optimal growth rate of consumption for an agent with income $y$ and the aggregate per capita consumption growth rate are:

$$\gamma_c(y) = (1/\theta) \cdot [A \cdot (1 - \tau'(y)) - \rho - \delta]$$

(28)

$$\gamma_c = (1/\theta) \cdot (A - \rho - \delta) - (A/\theta) \cdot (1/C) \cdot \int_0^\infty \phi(y) \cdot c(y) \cdot \tau'(y) \cdot dy$$

(29)

where $C$ is the level of per capita consumption, $c(y)$ is the optimal level of consumption chosen by each agent and $\phi(y)$ is the distribution of pre-tax income. If agents are homogeneous, then $\phi(y) = 1$ and $c(y) = C$ and thus (28) and (29) would collapse to (12).
Even though the transitional dynamics of (28) and (29) cannot be algebraically solved, they will follow the same pattern as those with homogeneous agents. Suppose that economies X and Z each possess two agents (rich and poor) and begin with the same initial distribution of income. As before, all agents have identical preferences and perfect access to capital markets. The tax system in both economies imposes higher tax rates on the wealthier agent to redistribute income to the poorer agent. Therefore, the after-tax return on capital is higher for the poorer agent. Tamura (1991) shows that in such a situation income levels will converge over time. This implies that each agent in steady state will be taxed at the highest rate.5

At time=0, the rich agent in economy X faces higher marginal tax rates, while the poor agent faces lower marginal tax rates than their respective counterparts in economy Z. In economy X, the rich agent accumulates less capital, while the poor agent accumulates more capital relative to economy Z. As a result, economy X experiences less total capital accumulation and thus lower transitional growth rates. Moreover, the income level of the poor agent in economy X rises faster over time due to greater capital accumulation and increased redistribution. Therefore, growth rates in economy X converge faster to their steady state.

5 As pointed out by one of the anonymous referees, this extension would yield the counterfactual prediction that all households would end up in the highest tax bracket in steady state. There are two ways to avoid this counterfactual result. First, as in Cassou and Lansing (1997a) (1997b), one could set total tax revenues as a fixed percentage of income. Therefore, during the transitional period, the tax structure would continually readjust downwards which would prevent all agents from entering the highest tax bracket. Second, one could change the assumptions on the households. Bovenberg and van Ewijk (1997) and Sarte (1997) find that combining nonlinear taxation with the assumption of overlapping generations or differing rates of time preference can maintain a non-degenerate distribution of income in steady-state. However, each of these modifications would prevent the transitional dynamics from being explicitly derived.
VII. CONCLUSION

This paper shows that the incorporation of a progressive tax structure into a linear growth model can generate both endogenous growth and convergence in per capita growth rates. As long as the marginal and average tax rates are sufficiently bounded from above, steady-state growth in capital and consumption can occur. Since capital accumulation brings about endogenous increases in tax rates, the after-tax average and marginal product of capital gradually decline along the transitional path. As a result, the transitional growth rates of consumption, capital and output fall. Furthermore, the paper found that an increase in tax progressivity through time reduces the transitional growth rates while maintaining the same steady-state growth rate. These results suggest that the tax structure or what is referred to as tax progressivity through time may be another source of differences between observed per capita growth rates.
REFERENCES


APPENDIX A

The first part of appendix A proves that the steady-state growth rates of consumption and capital must be of the same sign. Let \((\gamma_c)^*, (\gamma_k)^*\) and \((\gamma_y)^*\) represent the steady-state growth rates of consumption, capital and output, respectively.

Integration of the Euler equation (4) yields the following

\[
\lambda(t) = \lambda(0) \cdot \exp \left[ - \int_0^t \left( \rho + \delta - A \cdot (1 - \tau'(v)) \right) dv \right]. \quad (A1)
\]

Substituting (A1) into (5), the transversality condition becomes

\[
\lim_{t \to \infty} \left\{ k(t) \cdot \exp \left[ - \int_0^t \left[ A \cdot (1 - \tau'(v)) - \delta \right] dv \right] \right\} = 0. \quad (A2)
\]

According to (A2), the stock of capital cannot grow at a rate as high as the net marginal product of capital.

In steady-state, the capital accumulation equation (13) implies that

\[
c(t) = A \cdot k(t) - (1 - \xi) \cdot \tau(y) - k(t) \cdot (\gamma_k)^* . \quad (A3)
\]

By differentiating (A3) with respect to time, the relationship between consumption and capital growth in steady-state is

\[
\dot{c}(t) = \dot{k}(t) \cdot \left[ A \cdot (1 - \tau'_{ss}) + \xi \cdot A \cdot \tau'_{ss} - \delta - (\gamma_k)^* \right] \quad (A4)
\]

where \(\tau'_{ss}\) represents the steady-state marginal tax rate with respect to total income.

According to the transversality condition (A2), the term in the square brackets is positive. Therefore, \((\gamma_c)^*\) and \((\gamma_k)^*\) must be of the same sign. Furthermore, since output is a linear function of capital, \((\gamma_y)^*\) and \((\gamma_k)^*\) must be of the same sign and equal.
The second part of appendix A proves that (14) is a necessary and sufficient condition for steady-state growth. Consider the gross marginal of product of capital $A \in \mathbb{R}_{++}$. Figure 2 divides the range of $A$ into four intervals: 1, 2, 3, and 4.

**Proposition 1:** If $A \in 1$, then $[c(t), k(t), y(t)] = [0,0,0]$ for all $t > 0$. Since $\tau'(0) \geq \tau$, the net return on capital $A \cdot (1 - \tau'(0)) - \delta$ falls below the discount rate. In response, the agent will consume all of his initial net wealth.

**Proposition 2:** If $A \in 2$, then $(\gamma_c)^* = (\gamma_k)^* = (\gamma_y)^* = 0$.

To achieve steady-state growth, $(\gamma_k)^*$ is always positive when $k(t) \to \infty$ as $t \to \infty$. As $k(t) \to \infty$, the marginal tax rate approaches the upper bound $\bar{\tau}$ and thus $(\gamma_c)^* < 0$ by (12). This contradicts $(\gamma_c)^*$ and $(\gamma_k)^*$ being of the same sign. Moreover, as $k(t) \to \infty$ and thus $A \leq (\rho + \delta)/(1 - \tau)$, the transversality condition would fail which forces a downward jump in $k(t)$. If, on the other hand, $k(t) \to 0$, then $(\gamma_k)^* < 0$. However, as $k(t) \to 0$, then $(\gamma_c)^*$ approaches $(1/\theta) \cdot [A \cdot (1 - \tau) - \delta - \rho]$ which is positive and also contradicts $(\gamma_c)^* = (\gamma_k)^*$ being of the same sign. Therefore, the steady-state growth rates must be zero under proposition 2.

**Proposition 3:** If $A \in 3$, then $(\gamma_c)^* = (\gamma_k)^* = (\gamma_y)^* = (1/\theta) \cdot [A \cdot (1 - \tau) - \delta - \rho]$.

As $k(t) \to \infty$, $y(t) \to \infty$ and thus the marginal and average tax rates approach the upper bound $\bar{\tau}$. As a result, the growth rate of consumption $(\gamma_c)^*$ converges to $(1/\theta) \cdot [A \cdot (1 - \tau) - \delta - \rho]$ which is positive and independent of the state variable $k(t)$. Similarly, the growth rates $(\gamma_k)^*$ and $(\gamma_y)^*$ converge to $(1/\theta) \cdot [A \cdot (1 - \tau) - \delta - \rho]$. Therefore, under proposition 3, the sufficient conditions for steady-state growth are satisfied. The next proposition investigates the necessary conditions.

**Proposition 4:** If $A \in 4$, then $(\gamma_c)^* = (\gamma_k)^* = (\gamma_y)^* = 0$. 
Since the net return on capital exceeds the discount rate, then proposition 3 tells us that 
\[(\gamma_k)^* = (1/\theta) \cdot [A \cdot (1 - \tau) - \delta - \rho]\] if steady-state growth is to occur. In steady state, the transversality condition becomes 
\[A \cdot (1 - \tau) - \delta > (1/\theta) \cdot [A \cdot (1 - \tau) - \delta - \rho].\] This condition can be reduced to \[A < [\rho + \delta \cdot (1 - \theta)]/(1 - \tau) \cdot (1 - \theta)\] which is outside of interval 4. Alternatively, one can also show that if \(A \in 4\) there would be unbounded utility. Therefore, under proposition 4, the transversality condition is violated and no steady-state growth occurs.

According to propositions 1-4, the necessary and sufficient condition for steady-state growth in consumption, capital and output is
\[(\rho + \delta)/(1 - \tau) < A < [\rho + \delta \cdot (1 - \theta)]/[(1 - \tau) \cdot (1 - \theta)].\]  
This condition places joint restrictions on the parameters \(A, \rho, \delta, \) and \(\tau\).

**APPENDIX B**

Linearizing (19) and (20) around \(\chi^* = \phi\) and \(z^* = A \cdot (1 - \tau_0)\) gives:

\[
\begin{bmatrix}
\dot{z}(t) \\
\dot{x}(t)
\end{bmatrix} = \begin{bmatrix}
-(1-n) \cdot [z^* \cdot (1-\xi) - \chi^* + \xi \cdot A - \delta] & 0 \\
-\chi^* \cdot [(\theta \cdot (1-\xi) - n) / \theta] & \chi^*
\end{bmatrix} \begin{bmatrix}
(z - z^*) \\
(\chi - \chi^*)
\end{bmatrix}
\]

(B1)

Using the values for \(\chi^*\) and \(z^*\), the term \([z^* \cdot (1-\xi) - \chi^* + \xi \cdot A - \delta]\) equals the steady-state growth rate \(\gamma^*\):

\[\gamma^* = (1/\theta) \cdot [A \cdot (1 - \tau_0) - \delta - \rho].\]  
(B2)

The characteristic equation associated with (B1) is

\[\alpha^2 + \alpha \cdot [1-n \cdot \gamma^* - \chi^*] - (1-n) \cdot \gamma^* \cdot \chi^* = 0.\]  
(B3)

The characteristic equation (B3) has two roots:
\[ \beta_1 = -(1-n) \gamma^* < 0 \]  
\[ \beta_2 = \chi^* > 0. \]

Since one root is negative and the other is positive, the model follows a saddle-point path.

The general solution of (B1) is

\[ z(t) - z^* = c_1 \cdot A_1 \cdot \exp(\beta_1 \cdot t) + c_2 \cdot A_2 \cdot \exp(\beta_2 \cdot t) \]  
\[ \chi(t) - \chi^* = c_1 \cdot B_1 \cdot \exp(\beta_1 \cdot t) + c_2 \cdot B_2 \cdot \exp(\beta_2 \cdot t) \]

where \([ A_1, B_1 ]\) and \([ A_2, B_2 ]\) are the eigenvectors associated with the negative and positive roots, respectively and where \(c_1\) and \(c_2\) are positive constants of integration.

To rule out the positive root \(\beta_2\), one must show that all paths except for the stable arm violate at least one of the optimality conditions. Recall that \(z(t) \equiv [y(t) - \tau(y)] / k(t) = A \cdot (1 - \tau_0) + \tau_1 \cdot A^n \cdot k^{\alpha - 1}\) and \(\chi(t) \equiv c(t) / k(t)\). Consider a point that is below the stable arm in diagram 2. This point follows a path where \(\chi(t)\) falls to zero and \(z(t)\) falls to \(A \cdot (1 - \tau_0)\). As a result, \(c(t)\) falls to zero and \(k(t)\) approaches infinity. However, when \(c(t)\) falls to zero, then the costate variable \(\lambda(t)\) approaches infinity. Such a situation violates the transversality condition (5) and thus it cannot be optimal to begin below the stable arm.

Consider a point that is above the saddle point path in diagram 1. This point follows a path where both \(\chi(t)\) and \(z(t)\) approach infinity. As a result, \(c(t)\) grows without bound, while \(k(t)\) falls to zero in finite time. When \(k(t)\) reaches zero, then \(c(t)\) must jump from some positive number to zero. However, such a jump violates condition (12) and thus it cannot be optimal to begin above the stable arm. Therefore, \([ A_2, B_2 ]\) must equal \([0,0]\) for the optimality conditions (12), (13) and (5) to hold.
The constant of integration $c_1$ can be solved by setting $t = 0$ and substituting $z(0) = A \cdot (1 - \tau_0) + \frac{\tau_1 \cdot A^n \cdot k(0)^{n-1}}{[\phi - \beta_1]}$ into (B6):

$$c_1 = \frac{\tau_1 \cdot A^n \cdot k(0)^{n-1}}{[\phi - \beta_1]}$$  \hspace{1cm} (B8)

Substituting (B8) into (B6) and (B7) yields the particular solution for (B1)

$$z(t) - z^* = \tau_1 \cdot A^n \cdot k(0)^{n-1} \cdot \exp(\beta_1 \cdot t)$$  \hspace{1cm} (B9)

$$\chi(t) - \chi^* = \tau_1 \cdot A^n \cdot k(0)^{n-1} \cdot \phi \cdot \left[ \frac{\theta \cdot (1 - \xi) - n}{\theta \cdot (\phi - \beta_1)} \right] \cdot \exp(\beta_1 \cdot t).$$  \hspace{1cm} (B10)

These are equations (23) and (24) in the text.

Rewriting (12) and (13) in terms of $z(t)$ and $\chi(t)$, the transitional growth rates of consumption and output are

$$\gamma_c(t) = \gamma^* + \frac{1}{\theta} \cdot [z(t) - z^*]$$  \hspace{1cm} (B11)

$$\gamma_k(t) = \gamma^* + (1 - \xi) \cdot [z(t) - z^*] - [\chi(t) - \chi^*].$$  \hspace{1cm} (B12)

The linear production function implies that $\gamma_c(t) = \gamma_k(t)$. Substituting (B9) and (B10) into (B11) and (B12) gives equations (26) and (27) in the text.

**APPENDIX C**

The partial derivative of the transitional growth rate (27) with respect to the parameter $n$ is:

$$\frac{\partial \gamma_c(t)}{\partial n} = \tau_1 \cdot A^n \cdot k(0)^{n-1} \cdot \exp(\beta_1 \cdot t) \cdot \left[ \frac{\phi \cdot n + \theta \cdot (1 - \xi) \cdot (1 - n) \cdot \gamma^*}{\theta \cdot \phi - \theta \cdot (1 - n) \cdot \gamma^*} \right] \cdot \gamma^* +$$

$$\left[ \frac{\phi - \theta \cdot (1 - \xi) \cdot \gamma^*}{\theta \cdot \phi - \theta \cdot (1 - n) \cdot \gamma^*} \right] \cdot \theta \cdot \gamma^* +$$

$$\left[ \frac{\phi \cdot n + \theta \cdot (1 - \xi) \cdot (1 - n) \cdot \gamma^*}{(\theta \cdot \phi - \theta \cdot (1 - n) \cdot \gamma^*)^2} \right] \cdot \theta \cdot \gamma^*$$  \hspace{1cm} (C1)
\[
\left\{ \phi \cdot n + \theta \cdot (1-\xi) \cdot (1-n) \cdot \gamma^* \right\} \cdot \ln[Ak(0)]
\]

where equation (25) was substituted in for $\beta_1$. The first \[ term captures the change in the speed of convergence, while the last three \[ terms capture the change in the initial conditions. The last term depends directly upon the initial level of income and is non-negative as long as $k(0)$ is not too small.

For (C1) to be positive, the first and third terms must be greater than the second term. Let the fourth term equal zero. Through some algebraic manipulation, this condition can be expressed as

\[
(\phi)^2 + \gamma^* \cdot \phi \cdot [1 + \phi \cdot n \cdot t + n \cdot (1-n) \cdot t] + (\gamma^*)^2 \cdot \phi \cdot [\theta \cdot (1-\xi) \cdot (1-n) \cdot t]
\]

\[
(\gamma^*)^2 \cdot [\theta \cdot (1-\xi) \cdot (1-n) \cdot t] > \gamma^* \cdot \theta \cdot (1-\xi) \cdot \phi
\]

(C2)

Let $t = 1$ and ignore the higher-order growth terms. By using the definition of $\phi$ and the steady-state growth rate (22), condition (C2) can be reduced to

\[
[A \cdot (1-\tau_0) - \delta + \xi \cdot A \cdot \tau_0] + \phi \cdot n + n \cdot (1-n) > (1-\xi) \cdot [A \cdot (1-\tau_0) - \delta - \rho]
\]

(C3)

which holds for all parameter values. Therefore, an increase in the degree of progressivity $(1-n)$ lowers the transitional growth rate of output for all $t \geq 1$. 


Figure 1

\[ x = 0, \quad z = 0 \]

\[ x(0), \quad x^* \]

\[ z^*, \quad z(0) \]
Figure 2

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
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<tr>
<td>0</td>
<td>$\frac{\rho + \delta}{1 - \tau}$</td>
<td>$\frac{\rho + \delta}{1 - \bar{\tau}}$</td>
<td>$\frac{\rho + \delta \cdot (1 - \theta)}{(1 - \tau) \cdot (1 - \theta)}$</td>
<td></td>
</tr>
</tbody>
</table>